Assignment 11.

This homework is due *Thursday*, November 29.

Collaboration is welcome. If you do collaborate, make sure to write/type your own paper and *credit your collaborators*. Your solutions should contain full proofs. Bare answers will not earn you much. Extra problems is due December 7.

1. Quick reminder

Metric space is a pair (X, ρ), where X is a nonempty set and ρ is a function $\rho : X \times X \to \mathbb{R}$, called metric, such that $\forall x, y, z \in X$

(1) $\rho(x, y) \geq 0$,
(2) $\rho(x, y) = 0$ if and only if $x = y$,
(3) $\rho(x, y) = \rho(y, x)$,
(4) $\rho(x, y) \leq \rho(x, z) + \rho(z, y)$.

Normed linear space is a par $(V, \|\cdot\|)$, where V is a linear space and $\|\cdot\|$ is a funtion $\|\cdot\| : V \to \mathbb{R}$, called norm, such that $\forall u, v \in V$ and $\forall \alpha \in \mathbb{R}$,

(1) $\|u\| \geq 0$,
(2) $\|u\| = 0$ if and only if $u = 0$,
(3) $\|u + v\| \leq \|u\| + \|v\|$,
(4) $\|\alpha u\| = |\alpha| \|u\|$.

Every norm induces a metric via $\rho(u, v) = \|u - v\|$.

2. Exercises

(1) (9.1.4+)
  (a) Let $X = C[a, b]$. Show that $\|f\|_1 = \int_{[a,b]} |f|$ is a norm.
  (b) Show that the norm above is not equivalent to $\|f\|_{\max}$ (i.e. that there are no constants $c_1, c_2 > 0$ such that $\forall f \in C[a, b]$,
  $c_1\|f\|_1 \leq \|f\|_{\max} \leq c_2\|f\|_1$.)

(2) (∼9.1.5) Reminder: for sets A, B, their *symmetric difference* is defined as $A \triangle B = (A \setminus B) \cup (B \setminus A)$.
The Nikodym Metric. Let E be a Lebesgue measurable set of real numbers of finite measure. Let X be the set of Lebesgue measurable subsets of E, and m Lebesgue measure. For $A, B \in X$ define $\rho(A, B) = m(A \triangle B)$. Show that ρ is pseudometric, but not a metric, on X. Show that $\rho(A, B) = \int_E |\chi_A - \chi_B|$.

(3) Give an example of a metric on $\mathbb{R}$ not induced by any norm on $\mathbb{R}$.

(4) (a) (9.1.6) Show that for $a, b, c \geq 0$, if $a \leq b + c$, then $\frac{a}{1+a} \leq \frac{b}{1+b} + \frac{c}{1+c}$. (*Hint:* Straightforward way: multiply by common denominator; sneaky way: use concavity/convexity of $x/(1 + x)$.)
  (b) Let (X, ρ) be an arbitrary metric space. Prove that $(X, \frac{\rho}{1+\rho})$ is also a metric space.
  Note. This turns any metric space into a *bounded* metric space.
  (c) (9.1.10) Let $\{(X_n, \rho_n)\}$ be a countable collection of metric spaces. Show that ρ_* defines a metric space on the Cartesian product $\prod_{n=1}^{\infty} X_n$, where for points $x = \{x_n\}, y = \{y_n\} \in \prod_{n=1}^{\infty} X_n$,

$$\rho_*(x, y) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\rho_n(x_n, y_n)}{1 + \rho_n(x_n, y_n)}.$$

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(5) (9.2.20–22) For a subset E of a metric space X, a point $x \in X$ is called
- an interior point of E if there is $r > 0$ s.t. $B(x,r) \subseteq E$; the collection of interior points of E is called the interior of E and denoted $\operatorname{int} E$;
- an exterior point of E if there is $r > 0$ s.t. $B(x,r) \subseteq X \setminus E$; the collection of exterior points of E is called the exterior of E and denoted $\operatorname{ext} E$;
- a boundary point of E if for all $r > 0$, $B(x,r) \cap E \neq \emptyset$ and $B(x,r) \cap (X \setminus E) \neq \emptyset$; the collection of boundary points of E is called the boundary of E and denoted $\operatorname{bd} E$ or ∂E.

(a) Prove that $\operatorname{int} E$ is always open and that E is open iff $E = \operatorname{int} E$.
(b) Prove that $\operatorname{ext} E$ is always open and that E is closed iff $X \setminus E = \operatorname{ext} E$.
(c) Prove that $\operatorname{bd} E$ is always closed; that E is open iff $E \cap \operatorname{bd} E = \emptyset$; and that that E is closed iff $\operatorname{bd} E \subseteq E$.

(6) Let ρ and σ be two equivalent metrics on X.
(a) Prove that a sequence $\{x_n\}$ converges to x in (X, ρ) if and only if it converges to x in (X, σ).
(b) Prove that a subset $E \subseteq X$ is open in (X, ρ) if and only if it is open in (X, σ).
(*Hint:* Actually, (a)$\Leftrightarrow$(b), but proving that is about is much effort as proving them separately.)

3. Extra Problem

(7) Show that poitwise convergence in $C[0,1]$ is not metrizable. That is, show that there does not exist a metric ρ on $C[0,1]$ such that for $f_n, f \in C[0,1]$, a sequence $\{f_n\}$ converges pointwise to f if and only if $\lim_{n\to\infty} \rho(f_n, f) = 0$.

(8) Suppose X is a nonempty set and ρ, σ are two metrics on X. Suppose that a sequence $\{x_n\}$ in X converges to x in (X, ρ) if and only if it converges to x in (X, σ). Are ρ and σ are necessarily equivalent? (In other words, is converse to Problem 6a true?)